

THE REGULAR AND PROFINITE REPRESENTATIONS OF RESIDUALLY FINITE GROUPS

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ABSTRACT. Let G be a residually finite group. To any decreasing sequence $\mathcal{S} = (H_n)_n$ of finite index subgroups of G is associated a unitary representation $\rho_{\mathcal{S}}$ of G on the Hilbert space $\bigoplus_{n=0}^{+\infty} \ell^2(G/H_n)$. This paper investigates the following question: when does the representation $\rho_{\mathcal{S}}$ weakly contain the regular representation λ of G ?

INTRODUCTION

Let G be a countable residually finite group and H be a finite index, not necessarily normal, subgroup. The group G acts on G/H and we let $\lambda_{G/H}$ denote the representation that G admits on $\ell^2(G/H)$. That is, we let

$$\lambda_{G/H}(g)(\xi)(x) = \xi(g^{-1} \cdot x)$$

with $g \in G$, $\xi \in \ell^2(G/H)$ and $x \in G/H$. In this paper, we will mainly be interested in representations of the form

$$\rho_{\mathcal{S}} := \bigoplus_{n=0}^{+\infty} \lambda_{G/H_n}$$

where $\mathcal{S} = (H_n)_n$ is a decreasing sequence of finite index, not necessarily normal, subgroups of G . Such representations are called *profinite representations*. We will investigate the link between such representations and the regular representation λ of G . Recall that λ is the representation that G admits on $\ell^2(G)$ and defined for all $g \in G$, $\xi \in \ell^2(G)$ and $h \in G$ by

$$\lambda(g)(\xi)(h) = \xi(g^{-1}h).$$

As $\rho_{\mathcal{S}}$ is a sum of *finite representations* (i.e. factorizing through a finite index subgroup of G) and λ is a *C^0 -representation* (i.e. the coefficients $\langle \lambda(g)(\xi) | \psi \rangle$ tend to 0 when g tends to infinity), as soon as G is infinite, none of these two representations can be a subrepresentation of the other [Dix77]. However, one can expect a weaker link between them, namely weak containment. The goal of this work can be stated as follows: when does the representation $\rho_{\mathcal{S}}$ weakly contain the regular representation λ of G ?

Let us recall the definition of weak containment.

Key words and phrases. Residually finite groups, profinite representations, regular representation.

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Definition 0.1. If G is a countable group and (π_1, \mathcal{H}_1) , (π_2, \mathcal{H}_2) are two unitary representations of G , we say that π_2 weakly contains π_1 (and write $\pi_1 \prec \pi_2$) if and only if for all $\epsilon > 0$, $\xi \in \mathcal{H}_1$ and $K \subset G$ finite, there are vectors $\nu_1, \nu_2, \dots, \nu_n$ in \mathcal{H}_2 such that

$$(1) \quad \forall g \in K, \quad \left| \langle \pi_1(g)\xi | \xi \rangle - \sum_{i=1}^n \langle \pi_2(g)\nu_i | \nu_i \rangle \right| < \epsilon.$$

There is an equivalent formulation of this property in the framework of C^* -algebras [Dix77, Fel60]: for every unitary representation (π, \mathcal{H}) of G , we extend π linearly to get a $*$ -homomorphism

$$\ell^1(G) \xrightarrow{\pi} \mathcal{B}(\mathcal{H})$$

where $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded operators on \mathcal{H} . One then defines

$$C_\pi^*(G) := \overline{\pi(\ell^1(G))}$$

where the closure is taken with respect to the C^* -operator norm $\|\cdot\|_{\mathcal{B}(\mathcal{H})}$ on $\mathcal{B}(\mathcal{H})$. Then,

$$\begin{aligned} \pi_1 \prec \pi_2 &\iff C_{\pi_2}^*(G) \twoheadrightarrow C_{\pi_1}^*(G) \\ &\iff \forall f \in \ell^1(G), \quad \|\pi_1(f)\|_{\mathcal{B}(\mathcal{H}_1)} \leq \|\pi_2(f)\|_{\mathcal{B}(\mathcal{H}_2)} \\ &\iff \forall f \in \ell^1(G), \quad \text{sp}(\pi_1(f)) \subset \text{sp}(\pi_2(f)) \end{aligned}$$

where $\text{sp}(M)$ denotes the spectrum of a operator M . In particular, one remarks that π_1 and π_2 are weakly isomorphic (written $\pi_1 \simeq \pi_2$ and defined to be $\pi_1 \prec \pi_2$ and $\pi_2 \prec \pi_1$) if and only if the C^* -algebras $C_{\pi_1}^*(G)$ and $C_{\pi_2}^*(G)$ are isomorphic.

The representation ρ_S comes from an action of G on a countable set. In the preliminary Section 1, we introduce the notion of *locally somewhere free* action of a countable group G on a measure space (X, ν) :

Definition (1.1). We say that the action of G on (X, ν) is locally somewhere free (l.s.f. for short) if for every finite subset $K \subset G$, there is a Borel subset $F \subset X$ of positive measure such that the non-trivial elements of K fix almost no point in F :

$$\forall \gamma \neq 1 \in K, \quad x \in F \Rightarrow \gamma \cdot x \neq x \text{ a.e.}$$

If X is metrizable, locally compact and separable, and ν is a Radon measure quasi-preserved by G , then one can prove the following:

Proposition (1.2). If the action of G on X is l.s.f. then the regular representation λ of Γ is weakly contained in ρ_X .

Here, ρ_X denotes the canonical unitary representation that G admits on $L^2(X)$.

In order to use this result, we remark in Section 2 that there is a correspondence between profinite representations and actions on *rooted trees*. More precisely, to any decreasing sequence $\mathcal{S} = (H_n)_n$ of finite index subgroups of G , is associated a rooted tree $\mathcal{T}_{\mathcal{S}}$ together with a spherically transitive action of G on it. This correspondence is one-to-one and the representation $\rho_{\mathcal{S}}$ is isomorphic to the permutational representation $\rho_{\mathcal{T}_{\mathcal{S}}}$ coming from the action of G on $\mathcal{T}_{\mathcal{S}}$.

In the sequel, we mostly use this interpretation to adress our problem. We thus briefly recall in Section 3 some well-known facts about the automorphism group

$\text{Aut}(\mathcal{T})$ of a rooted tree. It turns out that the property $\lambda \prec \rho_{\mathcal{T}}$ is linked to the size of the stabilizers $\text{Stab}_G(v\mathcal{T})$ in G of the *subtrees* $v\mathcal{T}$ of \mathcal{T} . In Section 4, we use Proposition 1.2 of Section 2 in order to prove the following:

Theorem (4.1). *Let G be a countable group acting faithfully on a rooted tree \mathcal{T} .*

- i. If for every vertex $v \in \mathcal{T}^0$, the stabilizer $\text{Stab}_G(v\mathcal{T})$ is trivial, then $\lambda \prec \rho_{\mathcal{T}}$.*
- ii. More generally, if the set $\bigcup_{v \in \mathcal{T}^0} \text{Stab}_G(v\mathcal{T})$ is finite and has cardinality n , then $\lambda \prec \rho_{\mathcal{T}}^{\otimes n}$.*
- iii. One always has $\lambda \prec \bigoplus_{n=1}^{+\infty} \rho_{\mathcal{T}}^{\otimes n}$.*

The proof of the first part amounts to show that the triviality of the stabilizers $\text{Stab}_G(v\mathcal{T})$ is equivalent to the action of G on \mathcal{T} to be l.s.f., so that Proposition 1.2 can be applied. In Section 5, we study the inverse implication of this part of the Theorem.

Theorem (5.1). *Let G be a countable group in which the normalizer $N_G(H)$ of any non-central finite group H has infinite index in G .*

Suppose that G acts spherically transitively on a rooted tree \mathcal{T} . If there exists a subtree $v\mathcal{T}$ whose stabilizer $\text{Stab}_G(v\mathcal{T})$ in G is not trivial, then the $$ -homomorphism $\rho_{\mathcal{T}}$ defined on $\mathbb{C}G$ is not faithful. In particular, $\lambda \not\prec \rho_S$.*

Here, we have to make an algebraic assumption on G . Indeed, the sufficient condition in Theorem 4.1.i is not necessary in general (see Example 5.6).

In the last section, we illustrate the results of the previous two. In particular, we show that for the following classes of group, any faithful and spherically transitive action on a rooted tree is l.s.f.:

- (1) torsion free Gromov hyperbolic groups,
- (2) uniform lattices in a connected simple real Lie group G with finite center and \mathbb{R} -rank 1,
- (3) irreducible lattices in a connected semisimple real Lie group with finite centre, no compact factor and \mathbb{R} -rank ≥ 2 .

Thus, if G belongs to one of these classes, then any faithful profinite representation of G weakly contains the regular. Here is an application:

Corollary (6.6). *Let Γ be non-elementary, torsion free, residually finite hyperbolic group. Let $S = (H_n)_n$ be a decreasing sequence of finite index subgroups of Γ such that the representation ρ_S is faithful.*

Let $M_F := \frac{1}{2|F|} \sum_{g \in F} g + g^{-1}$ be the Markov operator associated to a finite generating set F of Γ which does not contain 1. Then

$$\left[-\frac{1}{|F|}, \frac{\sqrt{2|F|-1}}{|F|} \right] \subset \overline{\bigcup_n \text{sp}(\lambda_{\Gamma/H_n}(M_F))}.$$

Finally, we study the case of *weakly branched* subgroups of the automorphism group $\text{Aut}(\mathcal{T})$ of a *regular* rooted tree \mathcal{T} . They form an interesting and wide class of residually finite groups which provides important examples (infinite, periodic and finitely generated groups [Ale72], finitely generated groups with intermediate growth [Gri84], amenable groups not belonging to the class SG [GŻ02, BV05], finitely generated groups with non-uniform exponential growth [Wil04b, Wil04a, Bar03]).

A direct consequence of their definition is that their action on \mathcal{T} is not l.s.f. Thus, the results of §6.1 imply that a weakly branched group cannot belong to one of the classes (1),(2) or (3) above. It seems to the author that this was not known.

Finally, we prove the following proposition, which in particular indicates that Theorem 4.1.iii is optimal:

Proposition (6.9). *Let G be a weakly branched subgroup of $\text{Aut}(\mathcal{T})$. For every $n > 0$, the $*$ -homomorphism $\rho_{\mathcal{T}}^{\otimes n}$ defined on $\mathbb{C}G$ is not faithful. In particular, the representation $\rho_{\mathcal{T}}^{\otimes n}$ does not weakly contain the regular λ .*

1. LOCALLY SOMEWHERE FREE ACTIONS AND WEAK CONTAINMENT OF THE REGULAR REPRESENTATION

Let G be a countable group. Let (X, ν) be a metrizable, locally compact and separable space endowed with a Radon measure ν , on which G acts. If the action is measurable and quasi-preserved ν , then it yields the following unitary representation:

$$\begin{array}{ccc} G & \xrightarrow{\rho_X} & \mathcal{U}(L^2(X)) \\ g & \longrightarrow & f \mapsto \rho_X(g)(f) \end{array}$$

where $\rho_X(g)(f)$ is defined by

$$\rho_X(g)(f)(x) = \left(\frac{dg^{-1}\nu}{d\nu}(x) \right)^{1/2} f(g^{-1} \cdot x)$$

with $\frac{dg^{-1}\nu}{d\nu}$ the Radon-Nikodym derivative.

Definition 1.1. We say that the action of G on (X, ν) is *locally somewhere free* (l.s.f. for short) if for every finite subset $K \subset G$, there is a Borel subset $F \subset X$ of positive measure such that the non-trivial elements of K fixe almost no point in F :

$$\forall \gamma \neq 1 \in K, x \in F \Rightarrow \gamma \cdot x \neq x \text{ a.e.}$$

Proposition 1.2. *If the action of G on X is l.s.f. then the regular representation λ of Γ is weakly contained in ρ_X .*

Proof. Let K be a finite subset of X and F a subset of X of positive measure such that the non-trivial elements of K fixe almost no point in F . Let us consider a distance d on X compatible with its topology and for each positive integer n , the following measurable set:

$$E_n := \{x \in F \mid \forall \gamma \neq 1 \in K, d(\gamma \cdot x, x) > 1/n\}.$$

By definition, $\nu(F \setminus \cup_n E_n) = 0$ and therefore, there is a positive integer k such that $\nu(E_k) > 0$. Since X is separable, there is ball $B(x, l)$ of radius $l < \frac{1}{4k}$ such that $U_K := B(x, l) \cap E_k$ has non-zero measure. It satisfies

$$(2) \quad \forall \gamma \neq 1 \text{ in } K, \gamma(U_K) \cap U_K = \emptyset.$$

Let us now prove that $\lambda \prec \rho_X$. We denote by δ_1 the Dirac function over the identity element of G . This is a cyclic vector for the regular representation λ , i.e. the family $\{\lambda(g)(\delta_1) = \delta_g \mid g \in G\}$ is total in $\ell^2(\Gamma)$. Therefore, by a result of Fell [Fel63], it is enough to check (1) in Definition 0.1 for $\xi = \delta_1$. As such, we consider K a finite subset of Γ and $U := U_K$ the measurable subset associated to K defined

previously. If we let $\chi_U \in L^2(X)$ be the characteristic function of U , then (2) implies that for $g \in K$

$$\langle \lambda(g)(\delta_1) \mid \delta_1 \rangle = \delta_{1,g} = \left\langle \rho_X(g) \left(\frac{\chi_U}{\nu(U)} \right) \mid \frac{\chi_U}{\nu(U)} \right\rangle$$

where $\delta_{1,g} = 1$ if $1 = g$ and 0 otherwise. \square

Example 1.3. The action of a non-elementary, torsion free Gromov hyperbolic group on its boundary $(\partial G, \nu)$ endowed with a Patterson-Sullivan measure ν fullfills all the conditions of the previous proposition (see [Coo93]). It is l.s.f. since each element in G admits exactly two fixed points in ∂G , and the measure ν has no atom.

We stress that a faithful action is not necessarily l.s.f.; the last chapter will give such examples.

2. ROOTED TREES

Let $\bar{d} = d_0, d_1, \dots, d_n, \dots$ be a sequence of integers with $d_i \geq 2$ for all i . We define the rooted tree $\mathcal{T}_{\bar{d}}$ as follows: $\mathcal{T}_{\bar{d}}$ is an infinite, locally finite tree endowed with the usual metric $dist$ carried by any graph, and for which:

- 1) there is a particular vertex \emptyset called the *root*,
- 2) the degree $\deg(v)$ of any vertex v depends only on its distance to the root and is more precisely given by

$$\deg(\emptyset) = d_0, \text{ and for } n = dist(\emptyset, v) \geq 1, \deg(v) = d_n + 1.$$

Let $\mathcal{T}_{\bar{d}}^0$ denotes the set vertices of $\mathcal{T}_{\bar{d}}$. If n is a non-negative integer, the set of vertices whose distance to the root equals n is called the n -th level of $\mathcal{T}_{\bar{d}}$ and is denoted by L_n . One can give $\mathcal{T}_{\bar{d}}$ a *planar graph* structure as follows: for every n , one labels each vertex of L_n by a finite sequence $i_1 i_2 \dots i_n$ with $i_j \in \{1, 2, \dots, d_j\}$ such that 2 vertices $i_1 i_2 \dots i_n \in L_n$ and $i'_1 i'_2 \dots i'_{n+1} \in L_{n+1}$ are connected by an edge if and only if $i_1 i_2 \dots i_n = i'_1 i'_2 \dots i'_n$. Every level L_n is then endowed with the lexicographic order.

Given any vertex v , one defines the *subtree* $v\mathcal{T}_{\bar{d}}$ to be the connected subgraph of $\mathcal{T}_{\bar{d}}$ whose vertices w are *descendants* of v (i.e. $v \in [\emptyset, w]$ where $[\emptyset, w]$ denotes the unique geodesic path linking \emptyset and w).

We refer to Fig. 1 for a less rigourous but more visual presentation of these definitions.

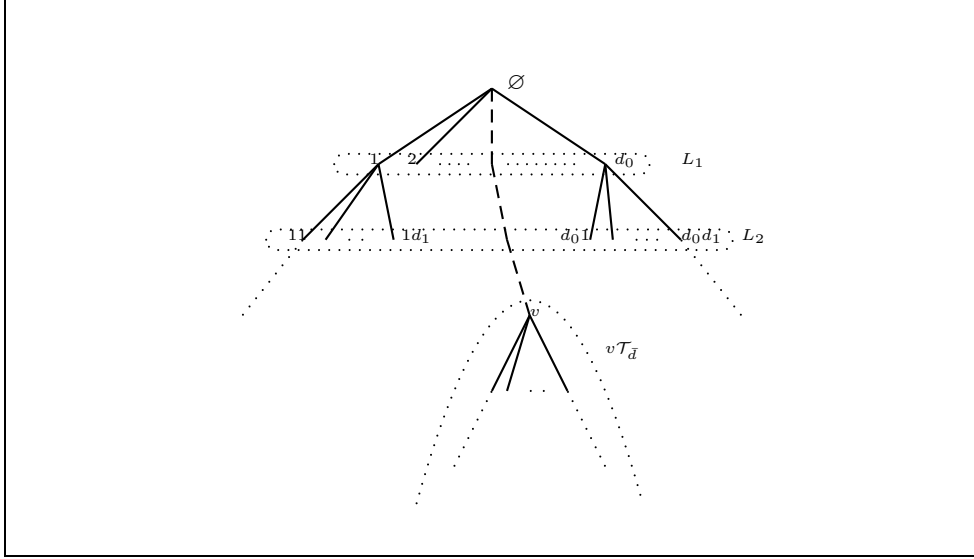
A countable group G is said to act on a rooted tree $\mathcal{T}_{\bar{d}}$ if it acts on the underlying graph while fixing the root. Such an action induces a unitary representation $\rho_{\mathcal{T}_{\bar{d}}}$ of G on the Hilbert space $\ell^2(\mathcal{T}_{\bar{d}}^0)$ defined by:

$$\forall g \in G, \forall \xi \in \ell^2(\mathcal{T}_{\bar{d}}^0), \forall v \in \mathcal{T}_{\bar{d}}^0, \rho_{\mathcal{T}_{\bar{d}}}(g)(\xi)(v) = \xi(g^{-1} \cdot v).$$

Let us also define, for every vertex $v \in \mathcal{T}_{\bar{d}}^0$ and every non-negative integer n the following stabilizers:

$$\text{Stab}_G(v) := \{g \in G \mid g(v) = v\}, \quad \text{Stab}_G(L_n) := \{g \in G \mid g(v) = v, \forall v \in L_n\}$$

which both have finite index in G , since G preserves the finite sets L_n . The representation $\rho_{\mathcal{T}_{\bar{d}}}$ will be faithful if and only if the action of G on $\mathcal{T}_{\bar{d}}$ is, that is if and only if $\bigcap_{n=0}^{\infty} \text{Stab}_G(L_n)$ is trivial.

FIGURE 1. Rooted tree \mathcal{T}

The action of G on $\mathcal{T}_{\bar{d}}$ is said *spherically transitive* if G acts transitively on each level. In this case, one has

$$\forall n, \forall v \in L_n, \text{Stab}_G(L_n) = \bigcap_{g \in G} g \text{Stab}_G(v) g^{-1}.$$

One notices that there is a correspondence between spherically transitive actions of G on a rooted tree $\mathcal{T}_{\bar{d}}$ and decreasing sequences $\mathcal{S} = (H_n)_n$ of finite index subgroups of G . Moreover, this correspondence respects the representations $\rho_{\mathcal{T}_{\bar{d}}}$ and $\rho_{\mathcal{S}}$.

Indeed, suppose G acts spherically transitively on $\mathcal{T}_{\bar{d}}$ and let $(v_n)_n$ be the sequence with v_n the *left most* vertex $11 \dots 1$ of L_n . Then, $\mathcal{S}(\mathcal{T}_{\bar{d}}) := (\text{Stab}_G(v_n))_n$ is a decreasing sequence of finite index subgroups of G fulfilling

$$\forall n, |\text{Stab}_G(v_n)/\text{Stab}_G(v_{n+1})| = d_n.$$

Moreover, since the actions of G on L_n and on $G/\text{Stab}_G(v_n)$ are isomorphic, the representations $\rho_{\mathcal{T}_{\bar{d}}}$ and $\rho_{\mathcal{S}(\mathcal{T}_{\bar{d}})}$ are isomorphic.

Conversely, let $\mathcal{S} = (H_n)_n$ be a decreasing sequence of finite index subgroups of G . Without loss of generality, one can assume that $H_0 = G$ and \mathcal{S} is strictly decreasing. Then, we can construct a rooted tree $\mathcal{T}(\mathcal{S})$ as follows:

- its vertices are the cosets $g_{n,k}H_n$,
- two vertices $g_{n,k}H_n$ and $g_{n',k'}H_{n'}$ are connected by an edge if and only if $n' = n + 1$ and $g_{n',k'}H_{n'} \subset g_{n,k}H_n$.

By construction, the root of $\mathcal{T}(\mathcal{S})$ is represented by G and, more generally, the sets L_n and G/H_n are equal: $\mathcal{T}(\mathcal{S})$ is actually the rooted tree $\mathcal{T}_{\bar{d}}$ where $\bar{d} = d_0, d_1, \dots$ is characterised by $d_n = |H_n/H_{n+1}|$. Moreover, it follows that G acts on L_n for all n , and thus on $\mathcal{T}(\mathcal{S})$. The representations $\rho_{\mathcal{S}}$ and $\rho_{\mathcal{T}(\mathcal{S})}$ are clearly isomorphic.

This correspondence implies for instance that a countable group G acts faithfully on a rooted tree if and only if it is residually finite. We will soon see that this picture is relevant for our purpose. More precisely, the weak containment of the regular representation λ in ρ_S , equivalently in $\rho_{\mathcal{T}}$, is linked to the size of the stabilizers $\text{Stab}_G(v\mathcal{T})$ of subtrees defined by

$$\forall v \in \mathcal{T}^0, \text{Stab}_G(v\mathcal{T}) := \{g \in G \mid g(x) = x, \forall x \in v\mathcal{T}\}.$$

Before starting the study of our problem, we recall in next section some convenient facts about the group of the automorphisms of a rooted tree.

3. THE AUTOMORPHISM GROUP OF A ROOTED TREE

By $\text{Aut}(\mathcal{T}_{\bar{d}})$ we denote the group of the automorphisms of $\mathcal{T}_{\bar{d}}$ that fix the root \emptyset . This short section aims to recall some convenient tools to describe such automorphisms. We refer to [Žuk08, Nek05, Nek09, Pla10] for more details.

For all n , the group $\text{Aut}(\mathcal{T}_{\bar{d}})$ preserves the level L_n as well as the finite set $\{v\mathcal{T}_{\bar{d}} \mid v \in L_n\}$ of subtrees rooted at its vertices.

Thanks to the self-similar structure of $\mathcal{T}_{\bar{d}}$, the group $\text{Aut}(\mathcal{T}_{\bar{d}})$ admits a natural decomposition in terms of the automorphisms group of other rooted trees.

More precisely, we have just noticed that every g in $\text{Aut}(\mathcal{T}_{\bar{d}})$ induces a permutation g_1 on the set L_1 , as well as an isomorphism $\varphi_{g(v)}(g)$ from $v\mathcal{T}_{\bar{d}}$ onto $g(v)\mathcal{T}_{\bar{d}}$, for every vertex v in L_1 .

These two subtrees are canonically isomorphic to $\mathcal{T}_{\sigma(\bar{d})}$ where σ denotes the *shift* on $\mathbb{R}^{\mathbb{N}}$ (i.e. $\sigma(\bar{d}) = d_1, d_2, \dots$) and $\varphi_{g(v)}(g)$ can be seen therefore as an element of $\text{Aut}(\mathcal{T}_{\sigma(\bar{d})})$.

It is easy to see that this data completely determines the action of g on $\mathcal{T}_{\bar{d}}$. In fact, we have the following decomposition:

$$\begin{array}{ccc} \text{Aut}(\mathcal{T}_{\bar{d}}) & \xrightarrow{\Phi} & \left(\text{Aut}(\mathcal{T}_{\sigma(\bar{d})}) \times \dots \times \text{Aut}(\mathcal{T}_{\sigma(\bar{d})}) \right) \rtimes \mathfrak{S}_{d_0} \\ g & \longrightarrow & (\varphi_1(g), \dots, \varphi_{d_0}(g)) \cdot g_1 \end{array}$$

where \mathfrak{S}_{d_0} denotes the symmetric group on the set of d_0 elements L_1 . Its action on $\left(\text{Aut}(\mathcal{T}_{\sigma(\bar{d})}) \times \dots \times \text{Aut}(\mathcal{T}_{\sigma(\bar{d})}) \right)$ is the permutation of the coordinates. The isomorphism Φ is called the *recursion isomorphism*.

Generalizing further, we denote by $\Phi^{(n)}$ the decomposition of $\text{Aut}(\mathcal{T}_{\bar{d}})$ with respect to the level L_n :

$$\begin{array}{ccc} \text{Aut}(\mathcal{T}_{\bar{d}}) & \xrightarrow{\Phi^{(n)}} & \left(\prod_{w \in L_n} \text{Aut}(\mathcal{T}_{\sigma^n(\bar{d})}) \right) \rtimes \text{Aut}(\mathcal{T}_{\bar{d},n}) \\ g & \longrightarrow & (\varphi_w)_{w \in L_n} \cdot g_n \end{array}$$

where $\mathcal{T}_{\bar{d},n}$ is the restriction of $\mathcal{T}_{\bar{d}}$ to its n -th first levels, and where $\text{Aut}(\mathcal{T}_{\bar{d},n})$ is the restriction of $\text{Aut}(\mathcal{T}_{\bar{d}})$ to this stable subgraph $\mathcal{T}_{\bar{d},n}$.

We remark that an element $g \in \text{Aut}(\mathcal{T}_{\bar{d}})$ fixes the restriction $\mathcal{T}_{\bar{d},n}$ if and only if g_n equals 1, which is also equivalent to g fixing the n -th level L_n .

4. A SUFFICIENT CONDITION

Let G be a countable group and $S = (H_k)_k$ a strictly decreasing sequence of finite index subgroups. To simplify the notations, let \mathcal{T} denote the associated rooted tree $\mathcal{T}(S)$. The goal of this section is to find a condition which ensures that ρ_S , or

equivalently $\rho_{\mathcal{T}}$, weakly contains the regular representation λ of G on $\ell^2(G)$. This is of course only possible if the representation ρ_S is faithful, i.e. if the action on \mathcal{T} is faithful.

What follows deals also with the representations $\rho_S^{\otimes n} \simeq \rho_{\mathcal{T}}^{\otimes n}$ that G admits on

$$\left(\bigoplus_{k=0}^{+\infty} \ell^2(G/H_k) \right)^{\otimes n} = (\ell^2(\mathcal{T}^0))^{\otimes n} = \ell^2(\mathcal{T}^0 \times \dots \times \mathcal{T}^0)$$

coming from the diagonal action of G on \mathcal{T}^n .

Theorem 4.1. *Let G be a countable group acting faithfully on a rooted tree \mathcal{T} .*

- i. *If for every vertex $v \in \mathcal{T}^0$, the stabilizer $\text{Stab}_G(v\mathcal{T})$ is trivial, then $\lambda \prec \rho_{\mathcal{T}}$.*
- ii. *More generally, if the set $\bigcup_{v \in \mathcal{T}^0} \text{Stab}_G(v\mathcal{T})$ is finite and has cardinality n , then $\lambda \prec \rho_{\mathcal{T}}^{\otimes n}$.*
- iii. *One always has $\lambda \prec \bigoplus_{n=1}^{+\infty} \rho_{\mathcal{T}}^{\otimes n}$.*

Proof of Theorem 4.1. i. Let us show that the action of G on the countable set \mathcal{T}^0 is l.s.f. so that Proposition 1.2 applies. If this is not the case, then there exists a finite subset F of G not containing 1 and such that $F \cap \text{Stab}_G(v)$ is non-empty, for all $v \in \mathcal{T}^0$. Let Υ be the map that associates to a vertex $v \in \mathcal{T}^0$ this non-empty finite set $F \cap \text{Stab}_G(v)$, and let $v_0 \in \mathcal{T}^0$ be such that $|\Upsilon(v_0)|$ is minimal. It is clear that if a vertex y is a descendant of a vertex x (i.e. $y \in x\mathcal{T}$), then $\Upsilon(y) \subset \Upsilon(x)$. By minimality of the cardinality of $\Upsilon(v_0)$ and the previous remark, we have

$$\forall v \in v_0\mathcal{T}, \quad \Upsilon(v) = \Upsilon(v_0)$$

and therefore any element in $\Upsilon(v_0)$, by construction necessarily non-trivial, fixes the subtree $v_0\mathcal{T}$. This conclusion contradicts the hypothesis of 4.1.i.

ii. Once again, we want to use Proposition 1.2 so suppose that F is a finite subset of G not containing 1. Let F_{free} be the subset of F consisting of its elements which do not fix any subtree:

$$F_{free} = F \setminus \left(F \cap \bigcup_{v \in \mathcal{T}^0} \text{Stab}_G(v\mathcal{T}) \right).$$

We set

$$\{h_1, h_2, \dots, h_k\} = F \setminus F_{free}.$$

For each h_i , there is an element $v_i \in \mathcal{T}^0$ such that

$$\text{Stab}_G(v_i) \cap (F_{free} \cup \{h_i\}) = \emptyset.$$

Indeed, let $w \in \mathcal{T}^0$ such that $h_i(w) \neq w$: by the definitions of w and F_{free} , no subtree of $w\mathcal{T}^0$ can be fixed by an element of $F_{free} \cup \{h_i\}$. We thus can apply the method of i. to prove the existence of such a v_i in the subtree $w\mathcal{T}^0$.

Now, we have an element $(v_1, \dots, v_k) \in (\mathcal{T}^0)^k$ such that

$$F \cap \text{Stab}_G(\{(v_1, \dots, v_k)\}) = F \cap \bigcap_{i=1}^k \text{Stab}_G(v_i) = \emptyset$$

where

$$k = |F \setminus F_{free}| = \left| F \cap \bigcup_{v \in \mathcal{T}^0} \text{Stab}_G(v\mathcal{T}) \right| \leq \left| \bigcup_{v \in \mathcal{T}^0} \text{Stab}_G(v\mathcal{T}) \right| = n.$$

Completing the sequence (v_1, \dots, v_k) by any vertices v_{k+1}, \dots, v_n , we get an element in $(\mathcal{T}^0)^n$ whose stabilizer in G does not intersect F . Therefore, the action on $(\mathcal{T}^0)^n$ is l.s.f. and Proposition 1.2 concludes.

iii. It is clear that for an action G on X , the action of G on $\bigsqcup_{n \in \mathbb{N}} X^n$ is l.s.f. as soon as the one on X is faithful. But $\bigoplus_{n=1}^{+\infty} \rho_{\mathcal{T}}^{\otimes n} = \rho_{\bigsqcup_{n \in \mathbb{N}} (\mathcal{T}^0)^n}$ so that Proposition 1.2 applies again. \square

Remark 4.2 (on the l.s.f. condition for actions on rooted trees). The first part of the proof shows that the triviality of all the stabilizers $\text{Stab}_G(v\mathcal{T})$ implies that the action is l.s.f. If the action is spherically transitive, then the converse holds. Besides, it is easy to see that if the stabilizer of an infinite geodesic path in \mathcal{T} is trivial, then the action is l.s.f. In particular, if $\mathcal{S} = (H_n)_n$ is a decreasing sequence of finite index subgroups of G with trivial intersection, then the action on the associated rooted tree $\mathcal{T}_{\mathcal{S}}$ is l.s.f. and the representation $\rho_{\mathcal{S}}$ weakly contains the regular. However, this condition is not necessary in general for the action to be l.s.f. (see for instance the realization of the *lamplighter group* $(\bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$ as an *automaton group* in [GŻ01]).

Remark 4.3. Notice that, for every positive integer n , $\rho_{\mathcal{S}}^{\otimes n}$ is a subrepresentation of $\rho_{\mathcal{S}}^{\otimes n+1}$. Indeed, the root of \mathcal{T} is fixed by G , therefore there are invariant vectors under G in $\ell^2(\mathcal{T}^0)$ and so the trivial representation ϵ is a subrepresentation of $\rho_{\mathcal{S}}$. It is thus not a surprise that the sufficient condition in 4.1.ii is weaker than the one in 4.1.i. In [Pla10] we give, for every positive integer n , some concrete examples of pair (G, \mathcal{S}) for which $\rho_{\mathcal{S}}^{\otimes 2^n}$ weakly contains λ , but $\rho_{\mathcal{S}}^{\otimes 2^n - 1}$ does not.

Remark 4.4. The statement 4.1.iii can be compared with the well known fact that if G is a finite group and ρ is a faithful representation of G , then there is a positive integer N such that every irreducible representation of G appears in $\rho^{\otimes N}$. Here we only get that the regular representation¹ is weakly contained in the sum of all the $\rho_{\mathcal{S}}^{\otimes n}$. But as we will see in the last section, there are examples of pair (G, \mathcal{S}) for which λ is not weakly contained in $\rho_{\mathcal{S}}^{\otimes n}$ for all n , and so the statement of 4.1.iii is optimal.

5. A NECESSARY CONDITION

In this section, we study the inverse implication of Theorem 4.1.i. It turns out to be true only under an algebraic assumption on G . After the proof of Theorem 5.1, we will give an example (see 5.6) of an action of a group on a rooted tree which shows that the sufficient condition of Theorem 4.1.i is not necessary in general and explains the additional assumption in Theorem 5.1 below.

Theorem 5.1. *Let G be a countable group in which the normalizer $N_G(H)$ of any non-central finite group H has infinite index in G .*

Suppose that G acts spherically transitively on a rooted tree \mathcal{T} . If there exists a subtree $v\mathcal{T}$ whose stabilizer $\text{Stab}_G(v\mathcal{T})$ in G is not trivial, then the $$ -homomorphism $\rho_{\mathcal{T}}$ defined on $\mathbb{C}G$ is not faithful.*

Remark 5.2. The conclusion of 5.1 clearly implies that the representation $\rho_{\mathcal{T}}$ does not weakly contain the regular representation λ of G ; indeed λ always defines a faithful representation of $\mathbb{C}G$. However, these two conclusions are in general not

¹which weakly contains every unitary representation of G if and only if G is amenable.

equivalent. Indeed, the regular representation of \mathbb{Z} extended linearly to $\mathbb{C}\mathbb{Z}$ is, via Fourier transform, given by multiplication on $L^2(S^1)$ by functions $S^1 \ni z \rightarrow \sum_{i=-l}^l \alpha_i z^i$ with $\alpha_i \in \mathbb{C}$. If one restricts this representation to $\mathcal{C}(\mathcal{A})$ where \mathcal{A} is a infinite closed strict subset of S^1 , one gets a representation which induces a faithful $*$ -homomorphism of $\mathbb{C}\mathbb{Z}$ (because a non-zero function $z \rightarrow \sum_{i=-l}^l \alpha_i z^i$ has finitely many zeros) and which cannot weakly contain the regular (because there are some functions $f = z \rightarrow \sum_{i=-l}^l \alpha_i z^i$ such that $\|f\|_\infty$ is only reached in $S^1 \setminus \mathcal{A}$).

Proof of Theorem 5.1. Let v be a vertex of the n -th level L_n of \mathcal{T} such that $\text{Stab}_G(v\mathcal{T})$ is not trivial. First note that such a group is not normal, thus non-central in G . Indeed, the G -action on the n -th level L_n is transitive, hence

$$\bigcap_{g \in G} g^{-1} \text{Stab}_G(v\mathcal{T}) g = \bigcap_{g \in G} \text{Stab}_G(g(v)\mathcal{T}) = \{1\}.$$

Moreover, this subgroup is normalized by the stabilizer $\text{Stab}_G(L_n)$ of the n -th level. The later having finite index in G , our additional assumption on G implies that $\text{Stab}_G(v\mathcal{T})$ is an infinite group. Therefore, it has a non-trivial intersection with the finite index subgroup $\text{Stab}_G(L_n)$.

Now, let us define for every subset A of L_n , the subgroup $\text{Stab}_G(L_n; A\mathcal{T})$ of G of the elements which fixe L_n as well as all the subtrees rooted at a vertex in A :

$$\text{Stab}_G(L_n; A\mathcal{T}) := \text{Stab}_G(L_n) \cap \bigcap_{a \in A} \text{Stab}_G(a\mathcal{T}).$$

The first paragraph of the proof concludes that $\text{Stab}_G(L_n; \{v\}\mathcal{T})$ is not trivial. So let A_0 be a subset of L_n such that $\text{Stab}_G(L_n; A_0\mathcal{T})$ is non-trivial, and such that among the subsets of L_n with this property, the cardinality of A_0 is maximal. Denote by (A_0, A_1, \dots, A_N) the orbit of A_0 under the G -action. One has:

Property 5.3. (A) $\bigcup_{i=0}^N A_i = L_n$ because G acts transitively on L_n .

(B) $\forall i = 1 \dots N, \exists \alpha_i \in G$

$$\text{Stab}_G(L_n, A_i\mathcal{T}) = \text{Stab}_G(L_n, \alpha_i(A_0)\mathcal{T}) = \alpha_i \text{Stab}_G(L_n, A_0\mathcal{T}) \alpha_i^{-1} \neq \{1\}.$$

(C) For every $i \neq j$, $g_i \in \text{Stab}_G(L_n; A_i\mathcal{T})$ and $g_j \in \text{Stab}_G(L_n; A_j\mathcal{T})$, one has

$$[g_i, g_j] := g_i g_j g_i^{-1} g_j^{-1} \in \text{Stab}_G(L_n; (A_i \cup A_j)\mathcal{T}) = \{1\}.$$

Therefore every element in $\text{Stab}_G(L_n; A_i\mathcal{T})$ commutes with every element in $\text{Stab}_G(L_n; A_j\mathcal{T})$.

The properties (A) and (B) are clear. Let us check (C). If we write down the decomposition of g_i and g_j with respect to the level L_n (see Section 3), we get

$$\begin{aligned} \Phi^{(n)}(g_i) &= (*, \dots, *, 1, *, \dots, 1, \dots) \\ \Phi^{(n)}(g_j) &= (*, \dots, 1, 1, *, \dots, *, \dots) \end{aligned}$$

where the 1s appear respectively in the positions corresponding to $x \in A_i$ and $y \in A_j$. Therefore

$$\Phi^{(n)}([g_i, g_j]) = (*, \dots, 1, 1, *, \dots, 1, \dots)$$

where the 1s appear in the positions corresponding to $z \in A_i \cup A_j$. Thus, we conclude that $[g_i, g_j]$ is an element of the group

$$\text{Stab}_G(L_n; (A_i \cup A_j)\mathcal{T})$$

which is by construction trivial since $|A_i \cup A_j| > |A_0|$.

Now, for all sequences g_0, g_1, \dots, g_N with $g_i \in \text{Stab}_G(L_n; A_i\mathcal{T})$, we define

$$M(g_0, g_1, \dots, g_N) = \prod_{i=0}^N (1 - g_i) \in \mathbb{C}G.$$

The following two lemmas clearly imply Theorem 5.1. \square

Lemma 5.4. *For every sequence g_0, g_1, \dots, g_N with $g_i \in \text{Stab}_G(L_n; A_i\mathcal{T})$,*

$$\rho_{\mathcal{T}}(M(g_0, g_1, \dots, g_N)) = 0.$$

Proof of Lemma 5.4. As we have already seen, the decomposition of the element $g_i \in \text{Aut}(\mathcal{T})$ with respect to the n -th level L_n is given by

$$\Phi^{(n)}(g_i) = (*, \dots, *, 1, *, \dots, 1, \dots)$$

where the 1s appear in the positions corresponding to $w \in E_i$. Thus, the operator $\rho_{\mathcal{T}}(1 - g_i)$ restricts to 0 on the invariant subspace

$$\left\{ f \in \ell^2(\mathcal{T}^0) \mid \text{supp}(f) \subset \bigcup_{k=0}^n L_k \cup \bigcup_{w \in E_i} w\mathcal{T} \right\}.$$

Because $\bigcup_{i=0}^N E_i = L_n$ (Property A), $\rho_{\mathcal{T}}(M(g_0, g_1, \dots, g_N)) = 0$. \square

Lemma 5.5. *There exists a sequence g_0, g_1, \dots, g_N with $g_i \in \text{Stab}_G(L_n; A_i\mathcal{T})$ such that*

$$M(g_0, g_1, \dots, g_N) \neq 0.$$

Proof of Lemma 5.5. The group $\text{Stab}_G(L_n; A_i\mathcal{T})$ is non-central in G and is normalized by the finite index subgroup $\text{Stab}_G(L_n)$. Hence, it is infinite by assumption. Two cases arise:

1st case: there exists an element $g_0 \in \text{Stab}_G(L_n; A_0\mathcal{T})$ whose order is infinite. The groups $\text{Stab}_G(L_n; A_i\mathcal{T})$ being conjugated (Property B), we can choose $g_i \in \text{Stab}_G(L_n; A_i\mathcal{T})$ of infinite order. These elements commute (Property C) and thus generate a group of the form $\mathbb{Z}^l \oplus K$ where K is a finite abelian group. If m is the cardinality of K , we have

$$H \stackrel{\text{def}}{=} \langle g_0^m, \dots, g_N^m \rangle = \mathbb{Z}^{l'}$$

with $l' \neq 0$ because, for instance, g_0^m has infinite order. It is well known that the algebra $\mathbb{C}H$ is a domain, thus $M(g_0^m, g_1^m, \dots, g_N^m) = \prod_{i=0}^N (1 - g_i^m) \neq 0$.

2nd case: the groups $\text{Stab}_G(L_n; A_i\mathcal{T})$ are periodic (and infinite). Let us construct inductively a sequence g_0, g_1, \dots, g_N such that:

- for all i , g_i is a non trivial element of $\text{Stab}_G(L_n; A_i\mathcal{T})$,
- for all i , $g_i \notin \langle g_0, \dots, g_{i-1} \rangle$.

The initial step of the induction is clear. Suppose g_0, \dots, g_i are already constructed, with $i < N$. The group $K = \langle g_0, \dots, g_i \rangle$ is generated by torsion elements which commute (Property C): K is a finite group and thus we can choose g_{i+1} in the infinite group $\text{Stab}_G(L_n; A_{i+1}\mathcal{T})$ which is not in K .

Now $M(g_0, \dots, g_N) = \prod_{i=0}^N (1 - g_i) \neq 0$. Indeed, expanding this product, one sees that its nullity implies a relation of the form

$$1 = g_{i_1} g_{i_2} \dots g_{i_l} \text{ with } i_1 < i_2 < \dots < i_l \text{ and } l \text{ odd.}$$

In particular it would follow that $g_{i_l} \in \langle g_{i_1}, \dots, g_{i_{l-1}} \rangle$ which is by construction impossible. \square

In the next example, we construct a group G together with a spherically transitive action on a rooted tree \mathcal{T} with non-trivial stabilizers of subtrees. These stabilizers are finite, non-central and normalized by a finite index subgroup of G . Thus, G is excluded from the framework of Theorem 5.1 and in fact, we will prove that the profinite representation of G on $\ell^2(\mathcal{T}^0)$ weakly contains the regular λ .

Example 5.6. On the finite set $X = \{1, 2, 3, 4, 5, 6\}$ (which will play the role of the first level L_1 of \mathcal{T}), consider the permutations

$$\alpha = (1, 3, 5)(2, 4, 6), \quad \beta_r = (1, 2)(3, 4).$$

Also let H be the group generated by α and β_r ; we denote by ρ_1 the permutational representation that H admits on $\ell^2(X) \simeq \mathbb{C}^6$ via its action on X .

Property 5.7. (A) The group K generated by β_r , $\beta_l := \alpha\beta\alpha^{-1}$ and $\beta_m := \alpha^2\beta\alpha^{-2}$ is a non-central normal subgroup in H and

$$H = K \rtimes \langle \alpha \rangle = (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/3\mathbb{Z}.$$

(B) The representation ρ_1 extended linearly to $\mathbb{C}H$ defines a faithful $*$ -homomorphism into $\mathcal{B}(\ell^2(X)) \simeq \text{End}(\mathbb{C}^6)$.

Proof. (A) One has

$$\beta_l = \alpha\beta\alpha^{-1} = (3, 4)(5, 6) \text{ and } \beta_m = \alpha^2\beta\alpha^{-2} = (1, 2)(5, 6)$$

so that $\beta\alpha\beta\alpha^{-1} = \alpha^2\beta\alpha^{-2}$. Therefore, the group $\langle \beta, \alpha\beta\alpha^{-1}, \alpha^2\beta\alpha^{-2} \rangle$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and is normal in H . Thus,

$$H = \langle \beta, \alpha\beta\alpha^{-1}, \alpha^2\beta\alpha^{-2} \rangle \rtimes \langle \alpha \rangle = (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/3\mathbb{Z}.$$

(B) We want to prove that every irreducible representation of H appears in ρ_1 . Let us first compute the character τ of the representation ρ_1 . Here, the map τ sends a permutation in H to the cardinality of its fixed points set:

$$\tau(h) = \begin{cases} 6 & \text{if } h = 1 \\ 2 & \text{if } h = \beta_l, \beta_m \text{ or } \beta_r \\ 0 & \text{otherwise.} \end{cases}$$

Let π be an irreducible representation of H whose character is denoted by ψ . We want to prove that $\langle \tau, \psi \rangle$ is non zero, i.e.

$$\frac{1}{|H|} \sum_{h \in H} \tau(h) \psi(h) \neq 0.$$

Thanks to the above computation, this is equivalent to

$$6\psi(1) + 2(\psi(\beta_r) + \psi(\beta_m) + \psi(\beta_l)) \neq 0.$$

Since the β_* are conjugate in G , their image under π have the same trace. Thus, we want to show (whatever $*$ is) that one of the following three equivalent statements is true:

$$\begin{aligned} 6(\psi(1) + \psi(\beta_*)) \neq 0 & \iff \dim \pi + \psi(\beta_*) \neq 0 \\ & \iff \pi(\beta_*) \neq -1. \end{aligned}$$

Now the last assertion is clear because $\beta_r \beta_l = \beta_m$ and it is impossible for the $\pi(\beta_*)$'s to all equal -1 . \square

Next consider the action of the group \mathbb{Z} on $\mathcal{T}_2 := \mathcal{T}_{2,2,\dots,2,\dots}$ given by the sequence of subgroups $\mathcal{S} = (2\mathbb{Z}, 4\mathbb{Z}, \dots, 2^n\mathbb{Z}, \dots)$. This action is generated by a single element; let $s \in \text{Aut}(\mathcal{T}_2)$ be this generator.

Before continuing our construction, we remark that Theorem 4.1.i and the remark 4.2 implies that $\rho_{\mathcal{S}}$ weakly contains the regular representation of \mathbb{Z} . The group \mathbb{Z} is amenable, hence these two representations are weakly isomorphic i.e.

$$(3) \quad C_{\rho_{\mathcal{S}}}^*(\mathbb{Z}) = C_{\rho_{\mathcal{T}_2}}^*(\mathbb{Z}) \simeq C_{\lambda}^*(\mathbb{Z}).$$

We now consider the rooted tree $\mathcal{T}_{6,2,2,\dots,2,\dots}$ whose first level L_1 is the set X on which H acts, and whose subtrees rooted at L_1 are all \mathcal{T}_2 on which \mathbb{Z} acts.

Definition 5.8. The group G is the subgroup of $\text{Aut}(\mathcal{T}_{6,2,2,\dots,2,\dots})$ generated by the elements $\bar{\alpha}$, $\bar{\beta}_r$ and \bar{s} defined via the recursion map (see Section 2):

$$\begin{aligned} \Phi(\bar{\alpha}) &= (1, 1, 1, 1, 1, 1) \alpha, \\ \Phi(\bar{\beta}_r) &= (1, 1, 1, 1, 1, 1) \beta_r, \\ \Phi(\bar{s}) &= (s, s, s, s, s, s). \end{aligned}$$

The subgroup $\langle \bar{\alpha}, \bar{\beta}_r \rangle$ is isomorphic to H and the element \bar{s} generates a copy of \mathbb{Z} in G which clearly commutes with H . Thus,

$$(4) \quad G = H \oplus \mathbb{Z}.$$

The group G acts spherically transitively on $\mathcal{T}_{6,2,2,\dots}$ because H acts transitively on its first level L_1 , and by construction \mathbb{Z} acts spherically transitively on all the subtrees \mathcal{T}_2 rooted at it. Moreover, the stabilizer of such a subtree is not trivial; for instance, the two right-most one (issued from the vertices labelled by 5 and 6) are fixed under the action of $\bar{\beta}_r$. The next proposition shows that the sufficient condition in Theorem 4.1.i is not necessary and explains the additional algebraic assumption in 5.1.

Proposition 5.9. The representation $\rho_{\mathcal{T}_{6,2,2,\dots}}$ of G on $\ell^2(\mathcal{T}_{6,2,2,\dots}^0)$ is weakly isomorphic to its regular λ .

Proof. To simplify notation, let us write ρ for $\rho_{\mathcal{T}_{6,2,2,\dots}}$. Maybe the easiest way to prove 5.9 is to use the language of C^* -algebras. We want to prove that

$$C_{\rho}^*(G) = C_{\lambda}^*(G).$$

We have

$$\begin{aligned} C_{\lambda}^*(G) &= C_{\lambda}^*(\mathbb{Z}) \otimes C_{\lambda}^*(H) \quad \text{by (4)} \\ &= C_{\lambda}^*(\mathbb{Z}) \otimes \mathbb{C}H \quad \text{because } H \text{ is finite.} \end{aligned}$$

Thus Proposition 5.9 is a consequence of (3) and the following lemma. \square

Lemma 5.10.

$$C_{\rho}^*(G) = C_{\rho_{\mathcal{T}_2}}^*(\mathbb{Z}) \otimes \mathbb{C}H$$

Proof of Lemma 5.10. Consider the restriction ρ' of ρ to the invariant subspace \mathcal{H} of $\ell^2(\mathcal{T}_{6,2,2,\dots}^0)$ consisting of functions null at the root of $\mathcal{T}_{6,2,2,\dots}$:

$$(5) \quad \mathcal{H} = \left\{ f \in \ell^2(\mathcal{T}_{6,2,2,\dots}^0) \mid \text{supp}(f) \subset \bigcup_{v \in L_1} v\mathcal{T}_{6,2,2,\dots} \right\}$$

$$(6) \quad = \bigoplus_{v \in L_1} \ell^2(\mathcal{T}_2^0).$$

The above restriction amounts to the removal of a copy of ϵ (the trivial representation) from ρ . But ρ' still contains ϵ since the constant functions on a level L_n belong to $\ell^2(L_n) \subset \mathcal{H}$. Therefore

$$C_\rho^*(G) = C_{\rho'}^*(G).$$

Now, the decomposition (6) of \mathcal{H} implies that $C_{\rho'}^*(G)$ is a subalgebra of $\mathcal{A} \otimes \text{End}(\mathbb{C}^6)$ where \mathcal{A} is the C^* -algebra generated by the restrictions $\varphi_v(g) \in \mathcal{U}(\ell^2(\mathcal{T}_2^0))$ with $v \in L_1$ and $g \in G$. By Definition 5.8 of the group G , the algebra \mathcal{A} is generated by s and hence is $C_{\rho\tau_2}^*(\mathbb{Z})$. Moreover, the only elements in G acting non-trivially on L_1 (i.e. inducing a non-trivial permutation of the factors $\ell^2(\mathcal{T}_2^0)$ in \mathcal{H}) are the elements of the subgroup H . Summarizing, we have

$$C_{\rho'}^*(G) = C_{\rho\tau_2}^*(\mathbb{Z}) \otimes \rho_1(\mathbb{C}H)$$

where ρ_1 is the permutational representation of H on $\ell^2(X)$ defined at the beginning of this paragraph, i.e. the restriction of ρ to $\ell^2(L_1)$. Lemma 5.10 is now a consequence of Property 5.7.B. \square

6. EXAMPLES AND APPLICATIONS

In this last section, we want to illustrate the results of the previous two. In particular, we will see that the sufficient condition of 4.1.i is automatically fulfilled for many lattices in semisimple real Lie groups, independently on the choice of a sequence \mathcal{S} defining a faithful representation $\rho_{\mathcal{S}}$. We will also see that for weakly branched groups, the situation is diametrically different.

6.1. Lattices in semisimple real Lie groups, hyperbolic groups.

6.1.1. Higher \mathbb{R} -rank case. The following proposition is a direct consequence of the Margulis Normal Subgroup Theorem [Mar78, Mar79, Zim84].

Proposition 6.1. *Let G be a connected semisimple real Lie group with finite centre, no compact factor and \mathbb{R} -rank ≥ 2 . Let Γ be an irreducible lattice in G acting faithfully and spherically transitively on a rooted tree \mathcal{T} . Then for all subtrees $v\mathcal{T}$ of \mathcal{T} , the stabilizer $\text{Stab}_\Gamma(v\mathcal{T})$ is trivial.*

Proof. Let v be a vertex of \mathcal{T} . The group $\text{Stab}_\Gamma(v)$ has finite index in Γ , hence is also an irreducible lattice in G . The stabilizer $\text{Stab}_\Gamma(v\mathcal{T})$ of the subtree rooted at v is a normal subgroup of $\text{Stab}_\Gamma(v)$. Moreover, $\text{Stab}_\Gamma(v\mathcal{T})$ has infinite index in Γ (because Γ acts spherically transitively on the subtree $v\mathcal{T}$). Therefore it has infinite index in $\text{Stab}_\Gamma(v)$. By the Margulis Normal Subgroup Theorem, $\text{Stab}_\Gamma(v\mathcal{T})$

is central in G and thus in Γ . From this follows that $\text{Stab}_\Gamma(v\mathcal{T})$ is trivial, because Γ acts transitively on L_n so that

$$\bigcap_{\gamma \in \Gamma} \gamma^{-1} \text{Stab}_\Gamma(v\mathcal{T}) \gamma = \{1\}.$$

□

6.1.2. Hyperbolic groups and \mathbb{R} -rank=1 case. The first result of this paragraph deals with Gromov hyperbolic groups. We refer to [Gro87, GdlH90] for details and proofs of the general facts which we use in the proof below.

Proposition 6.2. *Let Γ be a Gromov hyperbolic group. Assume that Γ is non-elementary and acts faithfully and spherically transitively on a rooted tree \mathcal{T} . Then $\bigcup_{v \in \mathcal{T}^0} \text{Stab}_\Gamma(v\mathcal{T})$ is finite.*

Remark 6.3. This result is optimal. Indeed, in Example 5.6, replace the 6 subtrees rooted at the first level on which \mathbb{Z} acts diagonally by 6 rooted trees $\mathcal{T}_{\vec{a}}$ on which a residually finite hyperbolic group Γ acts faithfully and spherically transitively. The subgroup of $\text{Aut}(\mathcal{T}_{6,\vec{a}})$ that we get is then $H \times \Gamma$ and is hyperbolic because H is finite. Here again, by construction, its action on $\mathcal{T}_{6,\vec{a}}$ is faithful and spherically transitive, but the subgroup H fixes the subtrees rooted at the two right-most vertices of the first level.

Proof. First, we prove that for every vertex $v \in \mathcal{T}$, the stabilizer $\text{Stab}_\Gamma(v\mathcal{T})$ is finite. If this is not the case, being a subgroup of a hyperbolic group, $\text{Stab}_\Gamma(v\mathcal{T})$ contains an element of infinite order γ . Let n be the level of v , i.e. $v \in L_n$; replacing γ by the non trivial element $\gamma^{N!}$ where $N = |L_n|$, we can assume also that γ fixes the n -th level L_n . Once again, we consider for every subset \mathcal{A} of L_n

$$\text{Stab}_\Gamma(L_n; \mathcal{A}\mathcal{T}) := \text{Stab}_\Gamma(L_n) \cap \bigcap_{a \in \mathcal{A}} \text{Stab}_\Gamma(a\mathcal{T}).$$

Above, we showed that $\text{Stab}_\Gamma(L_n; \{v\}\mathcal{T})$ is not trivial, and even contains a infinite order element. So consider \mathcal{A}_{\max} a subset of maximal cardinality for which $\text{Stab}_\Gamma(L_n; \mathcal{A}_{\max}\mathcal{T})$ contains an infinite order element, say γ . Of course, \mathcal{A}_{\max} is a strict subset of L_n . Hence, let us choose $v \in \mathcal{A}_{\max}$, $w \in L_n \setminus \mathcal{A}_{\max}$, and $\sigma \in \Gamma$ that such $\sigma(v) = w$ (this is possible because the action of Γ on L_n is assumed transitive).

Lemma 6.4. *There exists $h \in \text{Stab}_\Gamma(L_n)$ and integer k such that*

$$[\gamma^k, h\sigma\gamma^k\sigma^{-1}h^{-1}] := \gamma^k h\sigma\gamma^k\sigma^{-1}h^{-1}\gamma^{-k}h\sigma\gamma^{-k}\sigma^{-1}h^{-1}$$

has infinite order.

Proof. It is sufficient to prove that there is an $h \in \text{Stab}_\Gamma(L_n)$ such that

$$\text{Fix}_{\partial\Gamma}(\gamma) \cap \text{Fix}_{\partial\Gamma}(h\sigma\gamma\sigma^{-1}h^{-1}) = \emptyset$$

where $\text{Fix}_{\partial\Gamma}(\gamma) = \{\gamma^{+\infty}, \gamma^{-\infty}\}$ and $\text{Fix}_{\partial\Gamma}(h\sigma\gamma\sigma^{-1}h^{-1}) = \{h\sigma(\gamma^{+\infty}), h\sigma(\gamma^{-\infty})\}$ are the pairs of fixed points of γ and $h\sigma\gamma\sigma^{-1}h^{-1}$ in the boundary $\partial\Gamma$ of Γ . Indeed, it is known that in that case, the group generated by γ^k and $(h\sigma\gamma\sigma^{-1}h^{-1})^k = h\sigma\gamma^k\sigma^{-1}h^{-1}$ is the free group \mathbb{F}_2 , as soon as k is big enough.

If $\text{Fix}_{\partial\Gamma}(\gamma) \cap \text{Fix}_{\partial\Gamma}(\sigma\gamma\sigma^{-1}) = \emptyset$, we take $h = 1$. If this is not the case, we want to prove that $\text{Stab}_\Gamma(L_n)$ is not included in the subgroup P of G consisted of the elements preserving $\text{Fix}_{\partial\Gamma}(\gamma)$. This is easy because this last group contains

$\langle \gamma \rangle$ as a finite index subgroup, whereas $\text{Stab}_\Gamma(L_n)$ has finite index in Γ : as Γ is non-elementary, $\text{Stab}_\Gamma(L_n)$ is not amenable and thus cannot be a subgroup of P (which is quasi-isometric to \mathbb{Z}). \square

Fix h and k like in the statement of the previous lemma. Since γ^k belongs to $\text{Stab}_\Gamma(L_n; \mathcal{A}_{max}\mathcal{T})$ and h is in $\text{Stab}_\Gamma(L_n)$, the element $h\sigma\gamma^k\sigma^{-1}h^{-1}$ belongs to the group $\text{Stab}_\Gamma(L_n; \sigma(\mathcal{A}_{max})\mathcal{T})$. If we write down the decomposition of these two elements with respect to the level L_n , we get

$$\begin{aligned}\Phi^{(n)}(\gamma^k) &= (*, \dots, *, 1, *, \dots, 1, \dots) \\ \Phi^{(n)}(h\sigma\gamma^k\sigma^{-1}h^{-1}) &= (*, \dots, 1, 1, *, \dots, *, \dots)\end{aligned}$$

where the 1s appear respectively in the positions corresponding to $x \in \mathcal{A}_{max}$ and $y \in \sigma(\mathcal{A}_{max})$. Therefore

$$\Phi^{(n)}([\gamma^k, h\sigma\gamma^k\sigma^{-1}h^{-1}]) = (*, \dots, 1, 1, *, \dots, 1, \dots)$$

where the 1s appear in the positions corresponding to $z \in \mathcal{A}_{max} \cup \sigma(\mathcal{A}_{max})$. Lemma 6.4 then implies that $[\gamma^k, h\sigma\gamma^k\sigma^{-1}h^{-1}]$ is an infinite order element of the group

$$\text{Stab}_\Gamma(L_n; \mathcal{BT})$$

with $\mathcal{B} = \mathcal{A}_{max} \cup \sigma(\mathcal{A}_{max})$. This contradicts the maximality of $|\mathcal{A}_{max}|$ because $w \in \sigma(\mathcal{A}_{max}) \setminus \mathcal{A}_{max}$ and so $|\mathcal{A}_{max}| < |\mathcal{B}|$. We just proved that all the stabilizers $\text{Stab}_G(v\mathcal{T})$ are finite.

To complete the proof of 6.2, recall that in a hyperbolic group, there are only finitely many conjugacy classes of finite group. Hence, there exists a non-negative integer N such that

$$\forall k \in \mathbb{N}, \forall v \in L_{N+k}, \exists g \in G, i \leq N \text{ and } w \in L_i \text{ such that} \\ \text{Stab}_\Gamma(v\mathcal{T}) = g\text{Stab}_\Gamma(w\mathcal{T})g^{-1}.$$

As $g\text{Stab}_\Gamma(w\mathcal{T})g^{-1} = \text{Stab}_\Gamma(g(w)\mathcal{T})$, we conclude that

$$\bigcup_{v \in \mathcal{T}^0} \text{Stab}_\Gamma(v\mathcal{T}) = \bigcup_{i=1}^N \bigcup_{v \in L_i} \text{Stab}_\Gamma(v\mathcal{T})$$

is finite. \square

As a direct consequence of this proposition and Theorem 4.1.ii, we obtain:

Corollary 6.5. *Let Γ be non-elementary, residually finite hyperbolic group. Let \mathcal{S} be a decreasing sequence of finite index subgroups of Γ such that the representation $\rho_{\mathcal{S}}$ is faithful. Then, there exists a positive integer n such that the representation $\rho_{\mathcal{S}}^{\otimes n}$ of Γ weakly contains the regular λ .*

The next application shows how to use Proposition 6.2 to get information on the spectrum of Schreier graphs.

Corollary 6.6. *Let Γ be non-elementary, torsion free, residually finite hyperbolic group. Let $\mathcal{S} = (H_n)_n$ be a decreasing sequence of finite index subgroups of Γ such that the representation $\rho_{\mathcal{S}}$ is faithful.*

Let $M_F := \frac{1}{2|F|} \sum_{g \in F} g + g^{-1}$ be the Markov operator associated to a finite generating set F of Γ which does not contain 1. Then

$$\left[-\frac{1}{|F|}, \frac{\sqrt{2|F|-1}}{|F|} \right] \subset \overline{\bigcup_n \text{sp}(\lambda_{\Gamma/H_n}(M_F))}.$$

Proof. As M is self-adjoint, the closure of $\bigcup_n \text{sp}(\lambda_{\Gamma/H_n}(M_F))$ equals the spectrum of $\rho_S(M)$ and is a compact subset of \mathbb{R} . Now, the assumption that Γ is torsion free, Proposition 6.2 and Theorem 4.1.i together imply that ρ_S weakly contains the regular representation λ of Γ . In particular,

$$\text{sp}(\lambda(M_F)) \subset \text{sp}(\rho_S(M_F)) = \overline{\bigcup_n \text{sp}(\lambda_{\Gamma/H_n}(M_F))}.$$

Thanks to Proposition 5 in [dlHRV93], we know that $\text{sp}(\lambda(M))$ contains a pair $\{m, M\}$ with $m \leq -\frac{1}{|F|}$ and $M \geq \frac{\sqrt{2|F|-1}}{|F|}$. Since the Baum–Connes conjecture is true for hyperbolic groups [Laf02, MY02] and Γ is torsion free, it fulfills the Kadison–Kaplansky conjecture. Therefore, the spectrum $\text{sp}(\lambda(M))$ is connected. This proves Corollary 6.6. \square

We conclude with an analogous result to Proposition 6.1, in the case of uniform lattices in \mathbb{R} -rank 1 Lie groups.

Corollary 6.7. *Let Γ be a uniform lattice in a connected simple real Lie group G with finite center and \mathbb{R} -rank 1. Assume that Γ acts faithfully and spherically transitively on a rooted tree \mathcal{T} . Then for all subtrees $v\mathcal{T}$ of \mathcal{T} , the stabilizer $\text{Stab}_\Gamma(v\mathcal{T})$ is trivial and thus the representation $\rho_{\mathcal{T}}$ of Γ weakly contains the regular λ .*

Proof. Such a uniform lattice is hyperbolic. Let v be a vertex of \mathcal{T} . Proposition 6.2 implies that the stabilizer $\text{Stab}_\Gamma(v\mathcal{T})$ is a finite group. Therefore, its normalizer $N_G(\text{Stab}_\Gamma(v\mathcal{T}))$ in G is Zariski closed. Moreover, $\text{Stab}_\Gamma(v\mathcal{T})$ is a normal subgroup of $\text{Stab}_\Gamma(v)$; the later, having finite index in Γ , is also a lattice in G . By the Borel Density Theorem, the Zariski closure of $\text{Stab}_\Gamma(v)$ is G and therefore $\text{Stab}_\Gamma(v\mathcal{T})$ is a finite normal subgroup of G . Hence, $\text{Stab}_\Gamma(v\mathcal{T})$ has to be central in G , thus in Γ . Finally,

$$\text{Stab}_\Gamma(v\mathcal{T}) = \bigcap_{\gamma \in \Gamma} \gamma \text{Stab}_\Gamma(v\mathcal{T}) \gamma^{-1} = \{1\},$$

the last equality coming from the transitivity of the Γ -action on L_n . Theorem 4.1.i applies. \square

6.2. Weakly branched groups. This last section deals with natural examples of couple (G, \mathcal{T}) where G is a finitely generated group acting faithfully and spherically transitively on a rooted tree \mathcal{T} for which the conditions in Theorem 4.1.i and 4.1.ii are "far" from being true. These examples come from the class of *weakly branched* groups.

Definition 6.8. Let \mathcal{T} be a *regular* rooted tree, which means that $\mathcal{T} = \mathcal{T}_{\vec{d}}$ where \vec{d} is the constant sequence $\vec{d} = d, d, \dots, d, \dots$ with $d > 1$. A finitely generated subgroup G of $\text{Aut}(\mathcal{T})$ is said *weakly branched* if its action on \mathcal{T} is spherically transitive and

$$\forall v \in \mathcal{T}^0, \text{Rist}_G(v) := \bigcap_{w \in L_n \setminus \{v\}} \text{Stab}_G(w\mathcal{T})$$

is non-trivial.

The subgroup $\text{Rist}_G(v)$ consists of elements which only act on the subtree $v\mathcal{T}$. It is called the *rigid stabilizer* of v . It is easy to see that if G is weakly branched, these rigid stabilizers are infinite. So *a fortiori* are the stabilizers $\text{Stab}_G(v\mathcal{T})$ of subtrees.

We refer to [Žuk08, BGŠ03] for a survey on weakly branched groups. Concerning the link between $\rho_{\mathcal{T}}$ and λ for such a weakly branched group, one has:

Proposition 6.9. *Let G be a weakly branched subgroup of $\text{Aut}(\mathcal{T})$. For every $n > 0$, the $*$ -homomorphism $\rho_{\mathcal{T}}^{\otimes n}$ defined on $\mathbb{C}G$ is not faithful. In particular, the representation $\rho_{\mathcal{T}}^{\otimes n}$ does not weakly contain the regular λ .*

Proof. The method is the same that the one we used for the proof of Theorem 5.1. First, we see that remark 4.3 implies that we need only to prove the proposition for integers of the form $d^n - 1$. For every vertex v in the n -th level L_n of \mathcal{T} , choose g_v a non-trivial element in the infinite group $\text{Rist}_G(w)$. We have

$$(7) \quad \Phi^{(n)}(g_v) = (1, 1, \dots, 1, \varphi_v(g_v), 1, \dots)$$

where the non-trivial element $\varphi_v(g_v)$ appears in the position corresponding to v . It is clear that the g_v 's commute. Let

$$M = \prod_{v \in L_n} (1 - g_v) \in \mathbb{C}G.$$

Then, $M \neq 0$. Indeed, the nullity of M would imply the existence of a subset \mathcal{A} of L_n (of odd cardinality) such that

$$1 = \prod_{v \in \mathcal{A}} g_v,$$

and this is impossible because, if w is any vertex in \mathcal{A} , (7) implies that

$$\varphi_w \left(\prod_{v \in \mathcal{A}} g_v \right) = \varphi_w(g_w) \neq 1.$$

Let us show that $\rho_{\mathcal{T}}^{\otimes d^n - 1}(M)$ is 0. This is equivalent to proving that for every n -tuple $(z_1, \dots, z_{d^n - 1})$ consisting of elements in \mathcal{T}^0 ,

$$\rho_{\mathcal{T}}^{\otimes d^n - 1}(M) (\delta_{z_1} \otimes \dots \otimes \delta_{z_{d^n - 1}}) = 0$$

There is necessarily a vertex v_0 among the d^n in L_n such that the subtree $v_0\mathcal{T}$ does not contain any z_i . By construction,

$$\rho_{\mathcal{T}}^{\otimes d^n - 1}(1 - g_{v_0}) (\delta_{z_1} \otimes \dots \otimes \delta_{z_{d^n - 1}}) = 0$$

As the g_v 's commute,

$$M = \left(\prod_{v \in L_n \setminus \{v_0\}} (1 - g_v) \right) (1 - g_{v_0})$$

and this implies $\rho_{\mathcal{T}}^{\otimes d^n - 1}(M) (\delta_{z_1} \otimes \dots \otimes \delta_{z_{d^n - 1}}) = 0$. □

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